FOURIER ZOOM-IN AND WIDE RANGE ANGULAR SPECTRUM METHOD

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### Introduction (1)

**Rutherford–Sommerfeld equation**

The observed wave field can be calculated in terms of the incident wave field through aperture.

\[
U = -\frac{1}{4\pi} \int dS_1 U \frac{\partial}{\partial n} \left( \frac{e^{ikr_1}}{r_0} - \frac{e^{ikr_2}}{r_2} \right) = \int dS_1 U(P_1) \frac{e^{ikr_1}}{r_{01}} \left( \frac{1}{j\lambda} + \frac{1}{2\pi r_{01}} \right) \cos(n, r_{01})
\]

#### Numerical calculation

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*ASM: angular spectrum method
*FDM: Fresnel diffraction method

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Introduction (2)
: Bilinear interpolation

- Bilinear Interpolation
  - Interpolate the discrete source image
  - Artifacts (aliasing, blurring, edge halo) are removed with the extra aid of some filters.

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A. Ammar et al., 2012, Image zooming and multiplexing techniques based on K-Space transformation, J. Signal processing, image processing and pattern recognition, 5, 31.
Introduction (3):
Fourier zoom-in interpolation

- Fourier zoom-in interpolation
  - Increase the window in Fourier space and zero padding

(a) FFT + Zero padding
(b) Clipping

A. Ammar et al., 2012, Image zooming and multiplexing techniques based on K-Space transformation, J. Signal processing, image processing and pattern recognition, 5, 31.
Analysis of scaled ASM (1) : 3 step process

- Scaled angular spectrum method : 3 step
  1. Wave field of a source plane, \( g(x, y) \) is fast Fourier transformed to \( G(u, v) \).
  \[ G(m\Delta_u, n\Delta_v; 0) = \text{FFT}[g(k\Delta_{x, u}, l\Delta_{y, v}; 0)] \]

  2. Spatial frequency spectrum, \( G(u, v) \) propagates with a transfer function.
  \[ G(m\Delta_u, n\Delta_v; z) = G(m\Delta_u, n\Delta_v; 0) e^{2\pi i z ((m\Delta_u)^2 - (m\Delta_u)^2)} \]

  3. Spatial frequency spectrum, \( G(x, y) \) is inverse Fourier transformed to \( g(x, y, z) \).
  \[ g(k\Delta_{x, u}, l\Delta_{y, v}; z) = \frac{1}{M \times M} \sum_{m,n} G(m\Delta_u, n\Delta_v; z) \exp[j2\pi k\Delta_{x, u} m\Delta_u + j2\pi l\Delta_{y, v} n\Delta_v] \]

- If sampling intervals at a destination plane are different from that of a source plane, inverse Fourier transform have to be calculated by non-uniform fast Fourier transform (NUFFT) or by Chirp-Z transform.

Analysis of scaled ASM (2) : 5 step process

- Scaled angular spectrum method : 5 step
  - 1. Wave field of a source plane, \( g(x,y,0) \) is fast Fourier transformed to \( G(u,v,0) \).
    \[ G(mu,nu,0) = \text{FFT}(g(kx,ky,0)) \]
  - 2. Spatial frequency spectrum, \( G(x,y,0) \) propagate with a transfer function.
    \[ G(mu,nv,z) = \frac{G(mu,nu,0)}{1 - j2\pi \frac{M_y}{M_x} k_z \Delta x \Delta z} \]
  - 3. Wave field of a destination plane, \( g(x,y,z) \) with the same sampling interval.
    \[ g(kx,ky,z) = \text{IFFT}(G(mu,nu,z)) \]
  - 4. Fourier transform of wave field at a destination plane.
    \[ G(mu,nu,z) = \text{FFT}(g(kx,ky,z)) \]
  - 5. Spatial frequency spectrum, \( G(x,y,z) \) is inverse Fourier transformed to \( g(x,y,z) \).
    \[ g(kx,ky,z) = \frac{1}{2\pi M_y M_x} \sum_{m,n} G(mu,nu,z) \exp[j2\pi \frac{m k_x \Delta x}{M_x} + j2\pi \frac{n k_y \Delta y}{M_y}] \]
    
    \[ \frac{1}{M_y M_x} \sum_{m,n} G(mu,nu,z) \exp[j2\pi \frac{m k_x \Delta x}{M_x} + j2\pi \frac{n k_y \Delta y}{M_y}] \]
Analysis of scaled ASM (3)

: New interpretation

- Scaled angular spectrum method: 5 step
  1. Wave field of a source plane, \( g(x, y, 0) \) is fast Fourier transformed to \( G(k_{x}, k_{y}, 0) \).
  \[
  G(m_{x}, n_{y}, 0) = \text{FFT}(g(k_{x}, l_{y}, 0))
  \]
  2. Spatial frequency spectrum.
  \[
  g(m_{x}, n_{y}, k_{z}) = g(m_{x}, n_{y}, z)
  \]
  3. Wave field \( g(m_{x}, n_{y}, z) \) with the same sampling interval.
  \[
  g(k_{x}, l_{y}, z) = \text{FFT}(g(m_{x}, n_{y}, z))
  \]
  4. Fourier transform of wave field at a destination plane.
  \[
  G(m_{x}, n_{y}, z) = \text{FFT}(g(k_{x}, l_{y}, z))
  \]
  5. Spatial frequency spectrum, \( G(x, y, z) \) is obtained from \( G(m_{x}, n_{y}, z) \).
  \[
  g(k_{x}, l_{y}, z) = \sqrt{\frac{E_{0}}{2\pi}} \exp\left[\frac{-ik_{x}x}{M_{x}} + \frac{-ik_{y}y}{M_{y}}\right]
  \]

Wave field propagation

Angular spectrum method

Fourier zoom-in interpolation?
Fourier zoom-in interpolation (1): Equivalence proof to last page

- Fourier zoom-in interpolation
  - 1. Zero-padded spatial frequency spectrum, \( \tilde{G}_{zp}(u,v,z) \), is inverse Fourier transformed to Fourier zoom-in image, \( \tilde{g}_{zp}(k',l',z) \).

\[
\tilde{g}_{zp}(k',l',z) = \frac{1}{M_x M_y} \sum_{m=-M_x/2}^{M_x/2-1} \sum_{n=-M_y/2}^{M_y/2-1} \tilde{G}_{zp}(m \Delta_u, n \Delta_v, z) \exp \left[ -j2\pi \frac{mk' R_{sc}}{M_x} + j2\pi \frac{n l' R_{sc}}{M_y} \right]
\]

- 2. \( \tilde{G}_{zp}(u,v,z) \) is not zero when \((m, n)\) are between \((-M_x/2, -M_y/2)\) and \((M_x/2-1, M_y/2-1)\).

\[
\tilde{g}_{zp}(k',l',z) = \frac{1}{M_x M_y} \sum_{m=-M_x/2}^{M_x/2-1} \sum_{n=-M_y/2}^{M_y/2-1} \tilde{G}(m \Delta_u, n \Delta_v; z) \exp \left[ -j2\pi \frac{mk' R_{sc}}{M_x} + j2\pi \frac{n l' R_{sc}}{M_y} \right]
\]

- 3. Clip center region of interest \((k', l')\) are between \((-M_x/2, -M_y/2)\) and \((M_x/2-1, M_y/2-1)\).
Analysis of scaled ASM (3)

New interpretation

Scaled angular spectrum method: 5 steps

1. Wave field of a source plane, \( g(x, y, 0) \) is fast Fourier transformed to \( G(k_{x}, k_{y}, 0) \).
   \[ G(m_{x}a_{x}, m_{y}a_{y}, 0) = \text{FFT}(g(k_{x}a_{x}, k_{y}a_{y}, 0)) \]

2. Spatial frequency spectrum:
   \[ G(m_{x}a_{x}, m_{y}a_{y}) = \frac{G(m_{x}a_{x}, m_{y}a_{y}, 0)}{m_{x}a_{x}m_{y}a_{y}} \]

3. Wave field \( g(k_{x}a_{x}, k_{y}a_{y}, z) \) with the same sampling interval.
   \[ g(k_{x}a_{x}, k_{y}a_{y}, z) = \text{FFT}(G(k_{x}a_{x}, k_{y}a_{y}, z)) \]

4. Fourier transform of wave field at a destination plane.
   \[ G(m_{x}a_{x}, m_{y}a_{y}, z) = \text{FFT}(g(k_{x}a_{x}, k_{y}a_{y}, z)) \]

5. Spatial frequency spectrum, \( G(x, y, z) \) is interpolated to \( g(x, y, z) \).
   \[ g(k_{x}a_{x}, k_{y}a_{y}, z) = \frac{1}{m_{x}a_{x}m_{y}a_{y}} \sum_{m_{x}, m_{y}} G(m_{x}a_{x}, m_{y}a_{y}, z) \exp \left( \frac{j 2 \pi k_{x} m_{x} a_{x}}{N_{x}} + \frac{j 2 \pi k_{y} m_{y} a_{y}}{N_{y}} \right) \]
Fourier zoom-in interpolation (2): Numerical meaning

Complex field interpolation

\[
\hat{g}(k', l; z) = \frac{1}{M_x M_y} \sum_{m=0}^{M_y-1} \sum_{n=0}^{M_x-1} g(m, n) \exp \left[-2\pi i \left( k' l_{\text{ref}} - k n_{\text{ref}} - \frac{k' N_{\text{ref}}}{N_x} + \frac{m' R_{\text{ref}}}{R_x} - \frac{l' R_{\text{ref}}}{R_x} \right) \right] \sin \left[ 2\pi \left( k' l_{\text{ref}} - k n_{\text{ref}} - \frac{k' N_{\text{ref}}}{N_x} + \frac{m' R_{\text{ref}}}{R_x} - \frac{l' R_{\text{ref}}}{R_x} \right) \right] \sin \left[ 2\pi \left( \frac{k' R_{\text{ref}}}{R_x} - \frac{k' N_{\text{ref}}}{N_x} \right) \right]
\]

Rsc = 0.3

Graph showing the ratio for different values of \(k'\): \(k' = 0, 1, 2, 3\)
Fourier zoom-in interpolation (3): Variable zooming position

- Fourier zoom-in interpolation with variable zooming position
- 1. Region of interest around \((x_0, y_0)\) we can restrict variables \((k', l')\) between \((-M_x / 2 + x_0 / \Delta_{1x}, -M_y / 2 + y_0 / \Delta_{1y})\) and \((M_x / 2 - 1 + x_0 / \Delta_{1x}, M_y / 2 - 1 + y_0 / \Delta_{1y})\).

\[
g'(x', y', x_0, y_0; z) = g_{x_0}(x_0 + k' \Delta_{1x}, y_0 + l' \Delta_{1y}; x')
\]

\[
\sum_{m,n} C(m \Delta_u, n \Delta_v; z) \exp\left[\frac{j2\pi (m x + n y)}{M_x + n M_y}\right] = \sum_{m,n} C(m \Delta_u, n \Delta_v; z) \exp\left[\frac{j2\pi (m x + n y)}{M_x + n M_y}\right]
\]
Wide range ASM (1)
Band-limited ASM

- Nyquist theorem: local frequency of a function have to be below a half of a sampling frequency to avoid an aliasing error.

\[
\frac{1}{2\pi} \frac{\partial \omega}{\partial u} < \frac{1}{2\Delta u}, \quad \frac{1}{2\pi} \frac{\partial \omega}{\partial v} < \frac{1}{2\Delta v}
\]

- Angular spectrum method
- Band-limited ASM

Wide range ASM (2)
: Variable sampling interval

- Constant sampling interval in a Fourier space
  \[ \Delta_u = \frac{1}{2M_x \Delta_{1x}}, \Delta_v = \frac{1}{2M_y \Delta_{1y}} \quad G(0) = \text{FFT}\{g(0)\} \]

- Variable sampling interval in a Fourier space
  \[ \Delta_u = \frac{1}{R_{uv}(z)} \frac{1}{2M_x \Delta_{1x}}, \Delta_v = \frac{1}{R_{uv}(z)} \frac{1}{2M_y \Delta_{1y}} \quad G(0) = \text{NUFFT}\{g(0)\} \text{ or } G(0) = \text{Chirp Z} \{g(0)\} \]

---

Wide range ASM (3) : Accuracy vs. propagation distance

- The band limited ASM shows a maximum around $z = 80 \, S_1$ and falls below 10 dB when $z > 1000 \, S_1$.
- However, the wide range ASM keeps the increasing trend of PSNR until $z = 1585 \, S_1$ and remains above 45 dB up to $z = 100,000 \, S_1$.

Yong-Hae Kim et al., 2014, Non-uniform sampling and wide range angular spectrum method, J. Opt. 16 125710.
Simulation (1)

Geometry and PSNR

- Peak signal to noise:
  - $\text{Max}(I_{\text{rs}})$ is the maximum intensity obtained by the numerical integration of RS equation.
  - $\text{mean}([|g|^2 - I_{\text{rs}}])$ is the average of the absolute intensity difference between the simulation ($|g|^2$) and the numerical integration of RS equation ($I_{\text{rs}}$).

$$\text{PSNR} = 10\log_{10} \left( \frac{(\text{Max}(I_{\text{rs}}))^2}{\text{mean}([|g|^2 - I_{\text{rs}}])} \right)$$
Simulation (2): Zooming ratio variation

- PSNR is independent on the zooming ratio ($R_w$).
Simulation (3): Zooming ratio variation

- Maximum intensity is nearly independent to zooming ratio ($R_w$).

- RS integral
- Wide Range ASM
Simulation (4) : Zooming position variation

When the zooming position \((x_n)\) is increased, the PSNR is decreased at maximum to \(-20\) dB when propagation distance \(z\) is 1000 \(S_1\).
Simulation (5): Zooming position variation

- When the zooming position ($x_0$) is 0.4 $S_2$, the maximum intensity is about 10^{-2} of the intensity when zooming position ($x_0$) is 0 $S_2$.
Simulation (5)
: Zooming position variation

- When the zooming position \(x_0\) is 0.4 \(S_2\), the maximum intensity is about by \(10^{-2}\) of the intensity when zooming position \(x_0\) is 0 \(S_2\).
Simulation (5)

Zooming position variation

- When the zooming position ($x_0$) is 0.4 $S_2$, the maximum intensity is about 10^{-2} of the intensity when zooming position ($x_0$) is 0 $S_2$. 

![Graphs of intensity vs. position for different zooming positions](attachment:graphs.png)
Simulation (5):

:Zooming position variation

- When the zooming position \( x_0 \) is 0.4 \( S_2 \), the maximum intensity is about by \( 10^{-2} \) of the intensity when zooming position \( x_0 \) is 0 \( S_2 \).
Simulation (6)
:2-Dim. Lens effect

- Fourier zoom-in interpolation and wide range ASM show very similar intensify profile to that of direct integration of RS equation.

RS integral

Wide range ASM + Fourier zoom-in
Summary

- Analysis of scaled ASM
  - Wave field propagation using ASM
  - Fourier zoom-in interpolation

- Fourier zoom-in interpolation and wide range ASM
  - Accuracy is independent on zooming ratio.
  - Decrease of PSNR with the increase of zooming position can be explained by the decrease of the maximum intensity.
Thank you!